World Transactions on Engineering and Technology Education
Vol.11, No.4, 2013

Signal reconstruction - a project for students of electrical engineering

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ABSTRACT: A typical and important problem in telecommunications engineering is to reconstruct a signal function $x = x(t)$ if the phase of the Fourier transform $\hat{x} = \hat{x}(\omega)$ and some additional *a priori* information of convex type are given. How students can work on this special convex feasibility problem is described in this article. With some theoretical background, they can apply iterative solution methods already used in the algebraic approach of computer tomography. Since these methods contain some parameters, a lot of simulation experiments on a computer are necessary in order to choose good parameter values. The reconstruction needs some mathematical software, such as MATLAB. The students have to develop their own software parts. Using test signals, the numerical results can be evaluated and graphically demonstrated. The teams can compete for the best results, to be presented and discussed in a workshop at the end of the project phase.

INTRODUCTION

The aim of the project is to reconstruct a signal function $x = x(t)$ if the phase of the Fourier transform $\hat{x} = \hat{x}(\omega)$ and some additional *a priori* information of convex type are available. This is a well-known problem in telecommunications engineering. The problem can be presented to students, who should find efficient solution methods working in teams under the supervision of staff members (student centred project). The task catalogue can be as follows:

- Formulate the signal reconstruction problem (in continuous and discrete form).
- Study the corresponding theory and look for suitable reconstruction methods.
- Register the materials used (references from literature or the Web).
- Collect a set of signals, appropriate to test and train the methods (parameter adaptation).
- Develop software for reconstruction methods. Experiment with software files using the test signals.
- Study the influence of preconditions and parameters on the quality of reconstruction.
- Share your experiences with other team members and discuss further investigation also with the supervisor.
- Prepare a presentation or paper about the topic in cooperation with the team members.
- Indicate the contribution of the team and other sources used.

The project can be a good preparation for later work after graduation. Hence, the article presents some material for its realisation.

PROBLEM DESCRIPTION

It is assumed that the signals are *quadratically integrable* time functions $x = x(t)$ belonging to the Hilbert space $X = L_2(R)$. As an example, the signal:

$$x(t) = h(t) \cdot \exp(-t) \tag{1}$$

is considered, where *h(t)* is the unit step function. This signal starts with *x(0)=1* and decreases in time exponentially to *0*. The *frequency domain* of the signal is often used to reveal certain features important for its transfer, compression or reconstruction. It is given by the Fourier (integral) transform (FT):

$$\hat{x} = \hat{x}(\omega) = (Fx)(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} x(t) \cdot \exp(-j\omega t)\, dt \tag{2}$$

which is a (complex valued) function of frequency ω in $L_2(R)$. The symbol *j* represents the *imaginary unit* satisfying $j^2 = -1$. The polar representation of $\hat{x}$ reads:

$$\hat{x}(\omega) = r_x(\omega) \cdot \exp(\phi_x(\omega) \cdot j), \tag{2a}$$

where $r_x(\omega)$ is the *magnitude* or *amplitude* (spectrum) and $\phi_x(\omega)$ is the *phase* (spectrum) of the signal. In Example (1), the Fourier transform is given by:

$$\sqrt{2\pi} \cdot \hat{x}(\omega) = \frac{1}{1+\omega \cdot j} = \frac{1}{1+\omega^2} - \frac{\omega}{1+\omega^2} \cdot j = \frac{1}{\sqrt{1+\omega^2}} \cdot \exp\left(-\tan^{-1}(\omega) \cdot j\right). \tag{3}$$

Because of:

$$\left| \sqrt{2\pi} \cdot \hat{x}(\omega) - \frac{1}{2} \right| = \frac{1}{2}, \tag{3a}$$

the values of $\hat{x}$ lie on a circle in the complex plane. By Equation (3) amplitude and phase are:

$$r_x(\omega) = \frac{1}{\sqrt{2\pi(1+\omega^2)}}, \quad \phi_x(\omega) = -\tan^{-1}(\omega). \tag{3b}$$

The amplitude spectrum in Equation (3b) is a smooth even function with maximum $r_x(0)$ decreasing to *0* for increasing frequencies ω.

Assume that signal *x* is unknown but some *a priori* information is available. E.g. the phase $\phi(\omega)$ of *x* is known (by measurement). Then, *x* belongs to the closed convex set:

$$M_\phi = \left\{ x : \phi_x(\omega) = \phi(\omega) \ a.e. \right\}. \tag{4}$$

In Example (1), the phase is $\phi(x) = -\tan^{-1}(\omega)$. Further, perhaps the *carrier* of *x* (the closure of the set with nonzero values) is bounded or the *range* of *x* is restricted. Then, *x* is for appropriate closed intervals *I* and *J* an element of at least one of the two closed convex sets:

$$M_I = \left\{ x : x(t) = 0 \ a.e. \ for \ t \notin I \right\}, \quad M^J = \left\{ x : x(t) \in J \ a.e. \right\}. \tag{5}$$

In Example (1), e.g. *I = [0,10]* and *J = [0,1]* can be used. Referring to *I* this is only approximately true: the times *t>10* are cut, where *x(t)* is smaller then 0.0001.

Problem: A signal $x \in M = M_I \cap M_\phi$ or $x \in M = M_I \cap M^J \cap M_\phi$ has to be determined where $M \neq \emptyset$ is supposed.

This is a so-called *convex feasibility problem*. It can happen in practice that the intersection consists of more than one element *x* or is even inconsistent (empty). In the *regular case* of signal reconstruction, which is described by Hayes [1] and shortly repeated by Stark in Chapter 8 [2], the set *M* belongs to a one-dimensional subspace of L_2. Hence, possibly a scaled version of the original signal *x* is reconstructed. A further problem comes in by discretisation. One has to use discrete signals $x(t_k)$ with $t_0 < t_1 < ... < t_n$ instead of *x(t)*. Then, one has to apply the discrete Fourier transform (DFT) $\hat{x}(\omega_l)$ with $\omega_0 < \omega_1 < ... < \omega_n$ instead of the FT $\hat{x}(\omega)$. Finally, all work has to be done numerically on a computer using software. Hence, there are several sources for errors.

SOLUTION METHOD

A simple iterative solution method for huge and sparse systems of linear equations without typical pattern is called *projection onto hyperplanes* (PH). Geometrically, the single linear equations represent hyperplanes H_i (*i=1,2,…m*) in a Euclidean vector space *X*. The PH method projects step by step orthogonally onto these hyperplanes:

$$x_0 \in X, \quad x_{k+1} = P_k x_k = P\left(x_k \mid H_{i(k)}\right) = x_k + \Delta x_k, \tag{6}$$

where $i(k) \in \{1, 2, ..., m\}$ selects one of the H_i and P_k is the corresponding orthogonal projector onto this H_i. The update Δx_k is a normal of H_i. In the standard case, the selection is cyclic (see Figure 1a). The advantage of the PH method is that the projections can be easily determined. If the system of linear equations is consistent, that means, if the intersection of the hyperplanes is nonempty, one solution can be found by the PH method; namely, the solution which is nearest to the starting vector. It is only important that each hyperplane is selected again and again during the process. This method was proposed by Kaczmarz [3]. Later, it was applied and modified in computerised tomography (CT), introducing more general *relaxed* orthogonal projections:

$$T_k = P_k + \lambda_k (E - P_k) = \lambda_k E + (1-\lambda_k) P_k, \quad 0 < \lambda_k < 2 \tag{7}$$

as iteration operators, where E is the identity operator. This method:

$$y_0 \in X, \quad y_{k+1} = T_k y_k = y_k + \Delta y_k, \quad \Delta y_k = \lambda_k (P_k y_k - y_k) \tag{8}$$

was called *algebraic reconstruction technique* (ART) [4][5]. The *relaxation parameters* λ_k were used to accelerate the convergence and to control the limit. While for $\lambda_k = 1$, the original PH method arises, *overrelaxation* means a longer update ($1 < \lambda_k < 2$) and *underrelaxation* means a shorter update ($0 < \lambda_k < 1$). For a special case see Figure 1b.

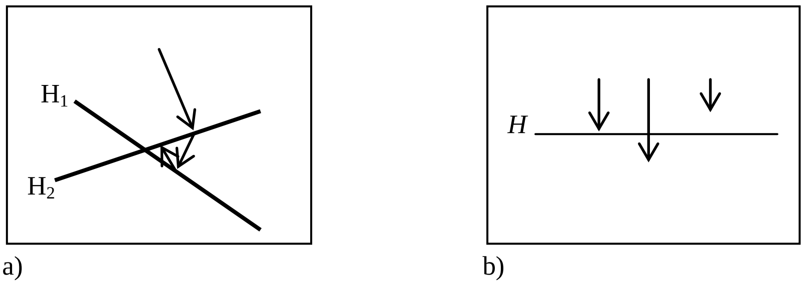

Figure 1: a) Successive orthogonal projections onto two straight lines H_1 and H_2 in the plane; b) Straight line *H* in the plane with orthogonal, overrelaxed and underrelaxed projections.

The orthogonal projector onto a hyperplane in a Euclidian vector space can be generalised to a *metric projector* onto a closed convex set *C* in a Hilbert space *X*. This projector $P = P(.,C)$ maps each element *x* onto the (uniquely determined) element $Px \in C$, which is the nearest to *x* in $C = F(P)$,the fixed point set of *P*:

$$\|x - Px\| \le \|x - c\| \quad \textit{for all } x \in X, c \in C. \tag{9}$$

The metric projector can again be relaxed by a parameter λ. More generally, *strongly Fejér monotone operators* $T = T(.,C,\alpha)$ with respect to *C* satisfying for certain $\alpha > 0$ the relations:

$$\alpha \|x - Tx\|^2 \le \|x - c\|^2 - \|Tx - c\|^2 \quad \textit{for all } x \in X, c \in C \tag{10}$$

can be used instead of projectors *P* while essentially conserving the convergence statements of the method. These operators *T* have also the fixed point set *C*. The projector *P* fulfils (10) with $\alpha = 1$, a relaxed projector belongs also to this class (10). The general convex feasibility problem assumes that there are convex closed sets C_i ($i = 1,2,\dots,m$) with nonempty intersection *C*. Then, the Fejér (projection like) method:

$$x_0 \in X, \quad x_{k+1} = T_k x_k = T\left(x_k, C_{i(k)}, \alpha_k\right) = x_k + \Delta x_k \tag{11}$$

converges to an element $x^* \in C$. Here $i(k) \in \{1,2,...,m\}$ selects one of the C_i and T_k is the corresponding strongly Fejér monotone operator with respect to C_i (compare (6)).

APPLICATION IN SIGNAL RECONSTRUCTION

First one considers iteration operators for the sets M_I and M_ϕ mentioned in the problem description. The set M^J is not used in this application. Relaxed projections for M_I have the form [6]:

$$\left(T_I x\right)(t) = q(t) \cdot x(t), \quad q(t) = \begin{Bmatrix} 1 & \textit{if } t \in I \\ 1-\lambda_I & \textit{if } t \notin I \end{Bmatrix}, \quad 0 < \lambda_I < 2. \tag{12}$$

This means $|q(t)| \leq 1$. Relaxed projections for M_ϕ are [6]:

$$\left(T_\phi^P x\right)(t) = F^{-1}(a_x \cdot r_x \cdot \exp(\phi \cdot j))(t), \quad a_x = \left(1-\lambda_\phi\right) \cdot \exp\left(\left(\phi_x - \phi\right) \cdot j\right) + \lambda_\phi \cos_+\left(\phi_x - \phi\right). \tag{13}$$

Hayes [1] used instead (with the cosine term replaced by 1):

$$\left(T_\phi^H x\right)(t) = F^{-1}(a_x \cdot r_x \cdot \exp(\phi \cdot j))(t), \quad a_x^H = \left(1-\lambda_\phi\right) \cdot \exp\left(\left(\phi_x - \phi\right) \cdot j\right) + \lambda_\phi \tag{13a}$$

Knowing the phase ϕ of a signal *x* (by measurement) and its carrier *I* (by a priori information), a corresponding signal has to be determined, that is, one is looking for $x \in M = M_I \cap M_\phi$. Hence, the two method classes:

$$m_1: \quad x_{k+1} = T_\phi T_I x_k, \quad m_2: \quad x_{k+1} = T_I T_\phi x_k \tag{14}$$

come in. Since m_2 can be interpreted as m_1 with modified starting signal one restricts to m_1. Special realisations are:

- HLO (with phase quasi-projection of Hayes [1]): method m_1 based on $T_\phi = T_\phi^H$ (13a).
- PCS (with relaxed phase projection): method m_1 based on $T_\phi = T_\phi^P$ (13).

In the experiments, the cyclic variant $z_{k+1} = T z_k, \quad T = T_\phi T_I$ with $z_0 \in X$ was applied.

TEST SIGNALS AND PARAMETERS

For experiments, a set of test signals was created, namely:

$$\begin{aligned} x^1(t) &= \frac{1}{2} \cdot \left(1 + \cos\frac{\pi \cdot t}{10}\right), \quad x^2(t) = \frac{1}{2} \cdot \left(1 - \frac{t}{50}\right) \cdot \left(1 + \cos\frac{\pi \cdot t}{10}\right) \\ x^3(t) &= \left|\cos\frac{t+6}{20} \cdot \sin\frac{t+6}{18}\right|, \quad x^4(t) = (t+5) \cdot \exp\left(-\frac{t}{20}\right) \end{aligned} \tag{15}$$

denoted in turn by T1, T2, T3 and T4. The starting signals for the methods were chosen from:

$$\begin{aligned} &x_0(t): \quad r_{x_0}(\omega) \equiv 1, \quad \phi_{x_0}(\omega) = \phi_x(\omega) = \phi(\omega) \\ &x_0(t) = \cos(t), \quad x_0(t) = 1 - \sin\left(\frac{t}{2}\right), \quad x_0(t) \equiv 1 \end{aligned} \tag{16}$$

denoted in turn by S1, S2, S3 and S4. The signal S1 is given in the frequency domain. It has already the right phase. The quality of approximations is influenced by the following preconditions: 1) signal (type) $x(t)$; 2) a priori information ϕ, *I* and *J*; 3) starting signal $x_0(t)$; 4) iterative method; 5) relaxation parameters λ_ϕ and λ_I; 6) step size *h* of discretisation; and 7) number *k** of iterations.

EXPERIMENTS

Signals $x(t) \in L_2[a,b]$ were discretised using *n* equidistant values $x(t_i) \in [a,b]$, where *i = 0,1,…,n* and $a = t_0$, $b = t_n$. It is necessary to add values $x(t_i) = 0$, where *i=n+1,…,N-1*. Here *N* is a power of *2* satisfying $N - 2 \geq 2n$. The extended interval $[t_0, t_{N-1}]$ has at least twice the length of [*a, b*]. So, a unique reconstruction is reached [1]. The Fourier transform (FT) is replaced by the discrete Fourier transform (DFT), using a fast realisation (FFT). The extended starting signal is iterated up to an approximation; finally, restricted again to [*a, b*]. This is in accordance with the fact that the DFT approximates the FT only in the first halve of the interval.

The tests used *a = 0*, *b = 50*, *n = 50* and *N = 128*. Choosing a step size of *h = 1* in the variable *t* the signals are represented by vectors $\vec{x}$ of length 128. Further, *k*=50* iteration cycles were carried out. Since the reconstruction is only unique up to a scalar c, the (rounded) relative errors (in the Euclidean norm) were related to:

$$x_{50}^*(t) = \frac{x(0)}{x_{50}(0)} \cdot x_{50}(t) \tag{17}$$

instead of $x_{50}(t)$ obtaining by this scaling the same initial value as the original *x(t)*. The smallest errors for each method are given in the following tables in italics.

Influence of relaxation parameters: In Table, 1 the signal $x(t) = x^2(t)$ is used as original starting point with $x_0(t) \equiv 1$. The table shows that high overrelaxation (both parameters in [1.5, 2]) causes the best results. Comparing the methods HLO seems to be slightly better than PCS, not only for this test signal, in contrary to the expectation. The cause that overrelaxation is successful can lie in a small angle between the linear hull of the restriction sets. The method PCS needs higher overrelaxation than HLO to get comparable quality of reconstruction.

Table 1: Influence of relaxation parameters.

Parameter 1	Parameter 2	HLO	PCS
0.5	0.5	1.2 E-1	1.7 E-1
1.0	1.0	5.9 E-3	9.7 E-3
1.5	1.0	5.2 E-4	1.1 E-3
1.0	1.5	6.1 E-4	1.1 E-3
1.5	1.5	*1.3 E-8*	7.7 E-8
1.7	1.7	1.3 E-8	*1.5 E-8*
2.0	2.0	3.0 E-1	2.7 E-1

Figure 2 presents the results for PCS and parameter values *1.7*:

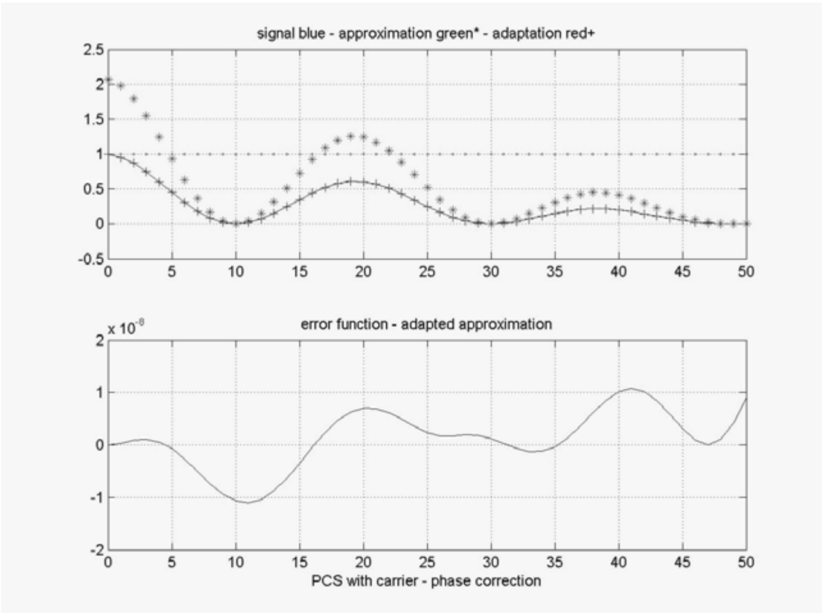

Figure 2: Reconstruction of signal T2 using overrelaxation.

Influence of starting signals: Again signal $x(t) = x^2(t)$ is used. Table 2 contains the relative errors after *k = 50* cycle steps for the standard parameter choice (both parameter values equals 1):

Table 2: Influence of starting signals.

Signal	HLO	PCS
S1	*2.6 E-3*	*3.7 E-3*
S2	3.7 E-3	6.3 E-3
S3	3.9 E-3	9.8 E-3
S4	5.9 E-3	9.7 E-3

It turns out that the starting signals are arranged by decreasing quality. The explanation seems to be simple. S1 already considers the given phase. S2 and S3 contain trigonometric functions as the original. S4 uses no additional information. Again the results of PCS can be made closer to HLO by choosing higher relaxation as HLO.

Influence of signal type: Using signals $x(t) = x^i(t)$ for *i = 1,2,3,4* and starting with signal S4 ($x_0(t) \equiv 1$). Table 3 gives the relative errors after *k*=50* cycle steps for the standard parameter choice (both parameter values equals 1):

Table 3: Influence of signal type.

Signal	HLO	PCS
T1	*5.3 E-3*	*7.7 E-3*
T2	5.9 E-3	9.7 E-3
T3	6.6 E-2	8.9 E-2
T4	8.9 E-2	1.1 E-1

The signal types are ordered by decreasing quality. The explanation seems to be not so simple here.

Further investigation: The given experimental data are not very representative. So, a lot of more experiments can be made, increasing the number of signals and starting signals, also investigating two-dimensional signals and adding other variants of the solution method. Possible are parameter optimisations in each cycle, weighted means of iteration operators, more general iteration operators, more a priori information, etc.

MATLAB RECONSTRUCTION FILES

The following MATLAB file realises the PCS algorithm. The file contains a fixed test signal and a fixed starting signal. It can be made more flexible if these signals must be specified after starting the file. The comments after the symbol % make the file self explaining.

```
% scriptfile PCS.m for a signal with restricted carrier reconstruction from phase
t = 0:127; ts = 0:50;                    % time interval, long and short form (carrier)
x = (0.5 + 0.5*cos(pi*t/10)).*(1-t/50);  % test signal 2
xa = x.*(t<=50); xs = x(1:51);           % short signal 2 (with and without zeros)
fx = fft(xa); afx = angle(fx);           % frequency domain, phase
y = ones(size(t)); y0 = y(1:51);         % special starting signal (identical 1)
disp(' '); K = input(' number of iteration cycles: '); disp(' ');
la_I = input(' relaxation parameter for carrier correction: ');
la_F = input(' relaxation parameter for phase correction: ');
for j = 1:K                              % iteration cycle steps
  ya = y.*(t<=50);                       % carrier correction
  yr = y + la_I*(ya-y);                  % relaxation 1
  fy = fft(yr); afy = angle(fy);         % frequency domain, phase
  fc = abs(fy).*exp(i*afx);              %  phase correction
  p  = cos(afy-afx); pp = (p>0).*p;      % cosine factor, positive part
  fp = pp.*fc;                           % FT of the projection
  fr = fy + la_F*(fp-fy);                % relaxation 2
  y  = ifft(fr);                         % inverse FT, iterated signal
end                                      % end of iteration
ys = real(y(1:51));                      % short approximated signal (real)
zs = x(1)/ys(1)*ys;                      % adaptation of initial value
ey = zs-xs; e = norm(ey)/norm(xs);       % relative error (vector norm)
disp(' '); disp(' relative error: '); disp(e);% output of relative error
subplot(2,1,1)                           % signal graphic, picture 1
plot(ts,xs,ts,ys,'*',ts,zs,'+',ts,y0,'.')  % plot of signals (see title)
title('signal blue - approximation green* - adaptation  red+')
subplot(2,1,2)                           % signal graphic, picture 2
plot(ts,ey)                              % plot of the error function
title('error function - adapted approximation')
xlabel('PCS with carrier - phase correction')
```

The excellent possibilities of MATLAB for graphic representations allow a vivid imagination of the approximation process and a detailed study of its local behaviour. The file reconstructs signal S2 with both parameter values *1.7* (see Figure 2). The file can be made even more user friendly by adding a graphical user interface (GUI).

CONCLUSIONS

The project is demanding and flexible but there is a task of practical value. A mathematical theory is needed to solve the task. Some mathematical background is already existing. However, further investigation is needed to obtain a correct solution. The numerical methods need software implementation on a computer. So, some experience from computer science and programming is necessary. A lot of experiments can be made to fit the methods to the task. A test set of signals is used to train the methods. Students can work in teams and can compete with other teams for the best results. The results can be presented at a workshop. Each presentation can be discussed and evaluated. At the end, a joint paper can be written. The project work can be used as input to later scientific work and graduation papers.

REFERENCES

1. Hayes, M.H., The reconstruction of a multidimensional sequence from the phase or magnitude of its Fourier transform, *IEEE Transactions on Acoustic Speech and Signal Process*, 30, 140-154 (1982).
2. Stark, H. (Ed), *Image Recovery: Theory and Application*s. New York: Academic Press (1987).
3. Kaczmarz, S., Angenäherte Auflösung von Systemen linearer Gleichungen. *Bull. Internat. Acad. Polon. Sci. Cl.* A, 35, 355-357 (1937).
4. Bender, R., Gordon, R. and Herman, G.T., Algebraic reconstruction techniques (ART) for three-dimensional electron microscopy and X-ray photography, *J. of Theoretical Biology*, 29, 471-481 (1970).
5. Censor, Y., Row-action methods for huge and sparse systems and their applications, *SIAM Rev*. 23, 444-464 (1981).
6. Schott, D., Signal reconstruction from given phase of the Fourier transform using Fejér Monotone methods. *Disc. Math., Diff. Incl. Contr. Optimiz.*, 20, 27-40 (2000).